ASYMPTOTIC CURVATURE OF MODULI SPACES FOR CALABI–YAU THREEFOLDS

THOMAS TRENNER AND P.M.H. WILSON

Abstract. Motivated by the classical statements of Mirror Symmetry, we study certain Kähler metrics on the complexified Kähler cone of a Calabi–Yau threefold, conjecturally corresponding to approximations to the Weil–Petersson metric near large complex structure limit for the mirror. In particular, the naturally defined Riemannian metric (defined via cup-product) on a level set of the Kähler cone is seen to be analogous to a slice of the Weil–Petersson metric near large complex structure limit. This enables us to give counterexamples to a conjecture of Ooguri and Vafa that the Weil–Petersson metric has non-positive scalar curvature in some neighbourhood of the large complex structure limit point.

Keywords Mirror symmetry, Weil–Petersson metric, large complex structure limit points, large radius limit points, curvature

Mathematics Subject Classification (2000) Primary 14J32, Secondary 32Q25, 53A15

Introduction

In this paper, we aim to understand the asymptotic behaviour of the Weil–Petersson metric near large complex structure limit points (defined in terms of maximally unipotent monodromy) on the complex moduli space of Calabi–Yau threefolds, by using a classical form of mirror symmetry and calculating the curvature of certain Kähler metrics on the complexified Kähler cone of the mirror.

This is intimately connected with the theory developed in [28]. If V is a Calabi–Yau threefold with $h^{2,0} = 0$ and $h^{1,1} = r$, we shall denote the cup-product cubic form on $H^2(V, \mathbf{R})$ by $f(y_1, \ldots, y_r)$. Let $\mathcal{K}(V) \subset H^2(V, \mathbf{R})$ denote the Kähler cone, and $\mathcal{K}_1 \subset \mathcal{K}(V)$ the level set given by $f = 1$. It is a consequence of the Hodge index theorem that the restriction of $-\frac{1}{6}(\partial^2 f/\partial y_i \partial y_j)$ to $\mathcal{K}_1$ defines a natural Riemannian metric on $\mathcal{K}_1$. In [28], it was argued in Section 1 that this Riemannian manifold reflects the asymptotic Weil-Petersson geometry on the moduli space of the mirror near large complex structure limit. This claim is given a more precise justification in this paper; in particular, see Remark 2.7 below.

The history of expectations concerning the curvature of the Weil–Petersson metric on the moduli space of Calabi–Yau threefolds is marked by unfulfilled hopes, probably over-influenced by the Weil–Petersson metric on the moduli space of curves, which was known to have negative curvature. In the Calabi–Yau threefold case, it was claimed in [23] that the holomorphic sectional curvatures were negative, and in [24] that all the sectional curvatures were negative. Both these

Date: 16 July 2010.

statements were disproved by the calculations of Candelas et al. [5] for the mirror quintic, where the 1-dimensional moduli space was shown to have Weil–Petersson curvature tending to $+\infty$ as one approaches the orbifold point.

The mirror quintic moduli space does however have negative curvature as one approaches the large complex structure limit point in moduli. This fact was generalised in [26], where it was shown that, whenever the complex moduli space of a Calabi–Yau threefold is 1-dimensional, the Weil–Petersson metric is asymptotic to the Poincaré metric near a large complex structure limit point, and in particular has negative curvature.

There was then a folklore expectation that this asymptotic negativity of the Weil–Petersson curvature near large complex structure limit should continue to hold for the complex moduli space having arbitrary dimension. This expectation motivated some of the work carried out in [28]. A weaker version of this expectation was articulated in Conjecture 3 of [20], where it was conjectured that at least the scalar curvature of the Weil–Petersson metric should be non-positive near the points at infinity in moduli (see also the evidence quoted in Example (v) in Section 3 of that paper). The second author pointed out in Section 2 of [29] that these expectations of asymptotic negativity were likely to be false, with conjectural counterexamples provided by the mirrors to the smooth Calabi–Yau Weierstrass fibrations over the Hirzebruch rational surfaces $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$, $\mathbf{F}_1$ and $\mathbf{F}_2$. In Section 3 below, we study in detail the case of the Weierstrass fibration over $\mathbf{F}_2$; this may also be conveniently described as the desingularization of a general hypersurface of degree 24 in $\mathbf{P}(1, 1, 2, 8, 12)$, and is a threefold known in the Physics literature as the STU-model. In Theorem 3.3, we see that the mirror to this Calabi–Yau threefold does indeed provide a counterexample to the conjecture in [20], since the Weil–Petersson scalar curvature is unbounded above in any neighbourhood of the large complex structure limit point. Moreover, we observe that the same thing happens for the mirrors to certain other toric hypersurface Calabi–Yau threefolds, and in Theorem 3.7 we extend this yet further to include a different type of Calabi–Yau threefold.

In Section 1 of this paper, we review various classical statements of mirror symmetry for Calabi–Yau threefolds. Motivated by these results, in Section 2 we study certain Kähler metrics on the complexified Kähler cone of a Calabi–Yau threefold, conjecturally corresponding to approximations to the Weil–Petersson metric near large complex structure limit for the mirror. In Section 3, we apply these general results to certain Calabi–Yau threefolds with $h^{1,1} = 3$, deducing results on the asymptotic Weil–Petersson geometry of their mirrors. In particular, we prove the existence of counterexamples to asymptotic non-positivity of the scalar curvature, both of the type predicted in [29] and of a different type.

The first author is supported on a studentship financed by the EPSRC, the Cambridge European Trust and a Trinity Hall Research Bursary. The second author wishes to thank Shinobu Hosono for the benefit of a useful conversation and email in December 2005.

1. Asymptotic Mirror Symmetry for Calabi–Yau threefolds

A classical formulation of the Mirror Symmetry Conjecture involves a *Mirror Map*, identifying a neighbourhood of a *large complex structure limit point* in the complex moduli space of a Calabi–Yau threefold with a neighbourhood of a *large radius limit point* in the Kähler moduli space of a mirror Calabi–Yau threefold,

under which the B-model correlation functions on complex moduli are identified with the A-model (quantum corrected) correlation functions on the Kähler moduli space of the mirror.

The Mirror Symmetry Conjecture stated in this form is carefully described in the book [6], in particular Chapters 5, 6, 7 and 8. Implicit in the statement is the idea of a *large complex structure limit point*, expressed in terms of a point of maximally unipotent monodromy (plus an integrality condition), or more precisely the choice of a simple normal crossing compactification of the complex moduli space around such a point. On the other side of the mirror, we have the idea of a *large radius limit point*, which involves a choice of smooth birational model V and a framing for its Kähler cone. A *framing* is a choice of an integral basis $T_1, \ldots, T_r$ of the torsion-free part of $H^2(V, \mathbf{Z})$ generating a simplicial cone σ in $H^2(V, \mathbf{R})$, whose interior $\mathrm{Int}(\sigma)$ is contained in the Kähler cone $\mathcal{K}(V)$. We shall denote by $\mathrm{im}(H^2(V, \mathbf{Z}))$ the image of the natural map $H^2(V, \mathbf{Z}) \to H^2(V, \mathbf{C})$. Throughout we assume that $h^{2,0}(V) = 0$, and so $\mathcal{K}(V)$ is an open convex cone in $H^2(V, \mathbf{R})$, and r denotes the Hodge number $h^{1,1}(V)$.

The *complexified Kähler cone* is defined by

$$\mathcal{K}_{\mathbf{C}}(V) = \{\omega \in H^2(V, \mathbf{C}) \ : \ \mathrm{Im}(\omega) \in \mathcal{K}(V)\}/\mathrm{im}(H^2(V, \mathbf{Z})),$$

the elements of this space usually being written as $B + i\omega$, where B is often called the *B-field*. The *complexified Kähler moduli space* is defined to be $\mathcal{K}_{\mathbf{C}}(V)/\mathrm{Aut}\,(V)$.

It is explained on pages 128-9 of [6] how a framing σ gives rise to a complex manifold

$$\mathcal{D}_\sigma = \{\omega \in H^2(V, \mathbf{C}) \ : \ \mathrm{Im}(\omega) \in \mathrm{Int}(\sigma)\}/\mathrm{im}(H^2(V, \mathbf{Z})),$$

and a biholomorphism from $\mathcal{D}_\sigma$ to $(D^*)^r \subset (\mathbf{C}^*)^r$ given by

$$t_1 T_1 + \ldots + t_r T_r \mapsto (q_1, \ldots, q_r) = (e^{2\pi i t_1}, \ldots, e^{2\pi i t_r}),$$

with $D \subset \mathbf{C}$ denoting the unit disc, and D^* the punctured disc. We may partially compactify $(D^*)^r$ to D^r, and the origin will then be referred to as a *large radius limit point*. For given class $B + i\omega \in \mathcal{D}_\sigma$, the limit of $B + it\omega$ as $t \to \infty$ is this large radius limit point. Having chosen the framing, we have uniquely defined local coordinates $q_1, \ldots, q_r$ on $\mathcal{D}_\sigma$, or equivalently coordinates $t_1, \ldots, t_r$ on its universal cover $H^2(V, \mathbf{R}) + i\,\mathrm{Int}(\sigma)$. Different framings of the Kähler cone will give rise to equivalent large radius limit points.

The first part of the Mirror Symmetry Conjecture says that to each large radius limit point of V, defined by a framing on the Kähler cone, there is a corresponding large complex structure limit point in the complex moduli of the mirror V°, and certain uniquely defined local coordinates $q_1, \ldots, q_r$ on an open neighbourhood of the large complex structure limit point, so that the Mirror Map identifies these coordinates q_i with those defined above on $\mathcal{D}_\sigma$ for V. For a given large complex structure limit point, we use the maximally unipotent monodromy condition to produce periods $y_i = \int_{\gamma_i} \Omega$ of the holomorphic 3-form Ω ($i = 0, 1, \ldots, r = h^{1,2}(V^\circ)$), with $\gamma_0, \gamma_1, \ldots, \gamma_r$ being part of a symplectic basis of $H_3(V^\circ, \mathbf{Z})$, where y_0 is holomorphic at the limit point and $y_1, \ldots, y_r$ have logarithmic singularities. The holomorphic period y_0 is well-defined up to a constant multiple, and the y_i are well-defined up to ordering and the addition of constant multiples of y_0. We may however always normalise the 3-form so that $y_0 = 1$; the integrality conjecture (briefly alluded to above) is needed in general to get a uniqueness statement for these periods, namely that the y_i for $1 \leq i \leq r$ are uniquely defined modulo ordering and the

addition of integral multiples of y_0. Assuming that y_0 has been normalised to be 1, the local coordinates required are $q_i = e^{2\pi i y_i}$; having permuted these coordinates appropriately, we should obtain the Mirror Map.

We now introduce the Yukawa couplings on the complex moduli side. For a given system of coordinates $z_1, \dots, z_r$, and choice of holomorphic 3-form Ω, we define the Yukawa couplings Y_{ijk} to be

$$Y_{ijk} = \int_{V^\circ} \Omega \wedge \nabla_{\partial/\partial z_i} \nabla_{\partial/\partial z_j} \nabla_{\partial/\partial z_k} \Omega,$$

or in some references (for instance [6]) with a minus sign in front, where ∇ denotes the Gauss–Manin connection. These couplings depend on the local coordinates chosen.

Assume now that we have a symplectic basis $A_0, A_1, \dots, A_r, B_0, B_1, \dots, B_r$ for $H_3(V^\circ, \mathbf{Z})$, and a holomorphic 3-form Ω with periods $\zeta_0, \zeta_1, \dots, \zeta_r$, $\xi_0, \xi_1, \dots, \xi_r$; a suitably general choice of A_0 ensures that the ratios $z_i = \zeta_i/\zeta_0$, for $1 \le i \le r$ form a local holomorphic coordinate system (the $\zeta_0, \dots \zeta_r$ are called *homogeneous special coordinates*) [4, 3]. If the corresponding dual basis for the torsion free part of $H^3(V^\circ, \mathbf{Z})$ is $\alpha_0, \alpha_1, \dots, \alpha_r, \beta_0, \beta_1, \dots, \beta_r$, the ζ_i and ξ_j are just the coordinates of the class represented by Ω with respect to the given basis. If we do not normalize Ω, then it is a consequence of theory from [4] that the ξ_j are holomorphic functions of the $\zeta_0, \dots, \zeta_r$, and that for some holomorphic function G of the ζ_i, we have $\xi_j = \partial G/\partial \zeta_j$ for all $j = 0, 1, \dots, r$ (see [3], pp 237-8). If we set $(\Omega, \bar\Omega) = -i \int \Omega \wedge \bar\Omega > 0$, it is checked easily that

$$(\Omega, \bar\Omega) = i \sum_j (\bar\zeta_j \partial G/\partial \zeta_j - \zeta_j \partial \bar G/\partial \bar\zeta_j) > 0.$$

We shall be interested in the *Weil–Petersson metric* on the complex moduli space; among various equivalent definitions, this may be defined as the Kähler metric with Kähler potential $-\log(\Omega, \bar\Omega)$, where the metric is readily seen to be independent of the choice of local holomorphic 3-form Ω. We wish to reinterpret the above in terms of *affine special coordinates* $z_i = \zeta_i/\zeta_0$, for $i = 1, \dots r$, local holomorphic coordinates on the moduli space. For details on these special coordinates, and derivations of the formulae below, see [22]. With this notation, $G(\zeta_0, \zeta_1, \dots, \zeta_r)/\zeta_0^2$ is a holomorphic function $F(z_1, \dots, z_r)$ of these coordinates, and the corresponding Yukawa couplings Y_{ijk} are given by $\partial^3 F/\partial z_i \partial z_j \partial z_k$; here F is sometimes called the *Gauss–Manin holomorphic prepotential*. In [6], this is called the Gauss–Manin potential, but we use the term prepotential to coincide with the usage from Physics [22, 10, 11], and to distinguish it from the potential function for the Kähler metric. With respect to these coordinates, one sees that the Weil–Petersson Kähler potential may be written as

$$K = -\log i\big(\sum_j (z_j - \bar z_j)(\partial_j F + \bar\partial_j \bar F) + 2\bar F - 2F\big),$$

where $\partial_j = \partial/\partial z_j$ and $\bar\partial_j = \partial/\partial \bar z_j$.

We explained above how, near a large complex structure limit point, we could find a well-defined set of periods $y_0, y_1, \dots, y_r$, on a neighbourhood $(D^*)^r$ of the limit point, where $D \subset \mathbf{C}$ is an open disc, with y_0 holomorphic on D^r and $y_1, \dots, y_r$ having logarithmic singularities. These gave rise to a holomorphic coordinate system $q_1, \dots q_r$ on some open neighborhood of the limit point in complex moduli; we

may however consider, within such a neighbourhood, small open sets in the (uncompactified) complex moduli space, on which therefore $y_0, y_1, \ldots, y_r$ may be considered as homogeneous special coordinates. Normalizing the 3-form Ω so that $y_0 = 1$, we obtain local affine special coordinates $y_1, \ldots, y_r$ — equivalently we may regard these as global coordinates on the universal covering of $(D^*)^r$. The corresponding Yukawa couplings Y_{ijk} are seen to be globally defined holomorphic functions on the neighbourhood $(D^*)^r$ of the limit point in complex moduli, although the holomorphic prepotential $F(y_1, \ldots, y_r)$ is well-defined only on the universal cover. The Yukawa couplings we have just defined are called the *normalized Yukawa couplings* (cf. [6], Definition 5.6.3, where they are taken with a negative sign), or the *B-model correlation functions*.

The other ingredients in classical mirror symmetry are (quantum corrected) Yukawa couplings on the Kähler side. These involve the Kähler class in the definition of the quantum corrections via Gromov–Witten invariants; the non-quantum part is given by the coefficients a_{ijk} in the cup-product cubic form

$$f(t_1, \ldots, t_r) = \sum_{i,j,k} a_{ijk} t_i t_j t_k,$$

with respect to the coordinates defined by the framing (see [6], Chapter 7). These also are called the *A-model correlation functions*. The *Mirror Symmetry Conjecture* says that, under the mirror map, the A-model and B-model correlation functions are identified.

The case where this can be described explicitly is that of Calabi–Yau hypersurfaces in 4-dimensional toric varieties $\mathbf{P}(\Delta)$ (where Δ is a reflexive polytope), or more precisely the Calabi–Yau threefolds obtained by an appropriate resolution of singularities. According to Batyrev's work [1], the mirror V° of such a Calabi–Yau hypersurface V is obtained by passing to the polar (sometimes called dual) reflexive polytope Δ°. We assume for simplicity that all deformations of the complex structure on V° are realised by deforming the defining polynomial of the hypersurface in $\mathbf{P}(\Delta^\circ)$; this last condition is equivalent to the property that, for any codimension 2 face Θ° of Δ°, with polar dual face $\hat{\Theta}^\circ$ of Δ, either Θ° or $\hat{\Theta}^\circ$ have no interior lattice points. Having chosen the desingularisation V of the hypersurface in $\mathbf{P}(\Delta)$, we can consider framings σ of its Kähler cone, where different framings then define equivalent large radius limit points. However, each such σ defines also a boundary point of the complex moduli space of V° (with σ interpreted as part of the GKZ decomposition, and hence also of the secondary fan). Conjecture 6.1.4 of [6] says that this boundary point is maximally unipotent (and that different framings give rise to equivalent boundary points), and the authors remark there that for threefolds the conjecture follows from an assertion of Givental [8]. In the case where the fan consisting of cones (with vertex at the origin) on the faces of Δ° may be subdivided at integral points on the faces of Δ° so as to achieve a regular fan (the polytopes of types I and II), this Conjecture was explicitly checked in Section 4 of [11] — see also (3.38) of [12].

The generators of the dual cone to σ determine natural toric coordinates to the complex moduli space at this limit, these coordinates being monomials in the coefficients of the defining polynomial. In order however to define the correct mirror map, we need to find local coordinates at the limit point defined as previously via periods, and we therefore need to calculate periods of the holomorphic 3-form.

One way to do this is to observe that they satisfy a certain set of generalised hypergeometric differential equations, known as the generalised GKZ system, and then to use a variant of the classical method of Frobenius to generate solutions from the (unique) holomorphic period (see [6, 11], and also (3.38) of [12]). In the toric hypersurface case, one can then define the mirror map uniquely without appealing to the integrality conjecture, and the toric mirror map so obtained is conjecturally the same as the mirror map one obtains via the integrality conjecture. The question of ordering the coordinates correctly is dealt with in the toric hypersurface case, as the derivative of the mirror map is the *monomial–divisor map* $H^{2,1}(V^\circ, \mathbf{C}) \to H^{1,1}(V, \mathbf{C})$, naturally produced by the toric machinery; for an explicit description of this, see Section 6.3.2 of [6]. As in the general case, we can define the normalized Yukawa couplings or B-model correlation functions, and the *Toric Mirror Symmetry Conjecture* says that, under the toric mirror map, the A-model and B-model correlation functions are identified. It was this form of the conjecture which was used in [10] to calculate (conjecturally) the Gromov-Witten invariants or instanton numbers of certain three-dimensional Calabi–Yau toric hypersurfaces, thus extending the famous calculations of Candelas et al. [5] for the quintic. The *Toric Mirror Symmetry Conjecture* is expected to hold for all toric hypersurface Calabi–Yau threefolds using the methods of Givental [8].

Both these classical mirror conjectures may be phrased in terms of holomorphic prepotentials. On the complex side, one has the Gauss–Manin prepotential $F(y_1, \dots, y_r)$ as defined above on the universal cover of an open neighbourhood of the limit point in complex moduli, a holomorphic function of $y_1, \dots, y_r$; on the Kähler side, one has the Gromov–Witten holomorphic prepotential defined on $H^2(V, \mathbf{R}) + i\,\mathrm{Int}(\sigma)$, as in Chapter 8 of [6]. Under the map identifying the complex coordinates y_i and t_i, the Gauss–Manin prepotential should be identified with the Gromov–Witten prepotential, modulo terms which are quadratic, linear or constant in the t_i (see [6], Corollary 8.6.3).

In this paper however, we shall only need asymptotic forms of these conjectures, that the mirror map is well-defined and the limit of the B-model correlation functions Y_{ijk} as one approaches a large complex structure limit point in complex moduli will be the coefficients a_{ijk} of the cubic cup-product form on the mirror (with respect to the coordinates determined by the corresponding framing). This may be rephrased as saying that the B-model correlation functions near a large complex structure limit point in complex moduli will correspond, under the (toric) mirror map, to the topological (uncorrected) coupling, plus a correction term which is holomorphic in the coordinates $q_j = e^{2\pi i t_j}$ and takes value zero at the large radius limit. If we consider the cup-product cubic form f as a cubic polynomial in the complex coordinates t_i on $H^2(V, \mathbf{R}) + i\,\mathrm{Int}(\sigma)$, then the asymptotic form of the conjecture says that under the (toric) mirror map, the Gauss–Manin holomorphic prepotential will correspond to $f(t_1, \dots, t_r)/6$, modulo terms which are quadratic, linear or constant, and a quantum correction term which is holomorphic in the q_i with no constant term, and therefore decays exponentially in the t_i near large radius limit.

The Asymptotic Toric Mirror Symmetry Conjecture was explicitly checked in Section 4 of [11] for toric Calabi–Yau hypersurfaces corresponding to reflexive polytopes with the property that all deformations are of polynomial type and which are of Type I or II. A precise summary of the relevant results is given on page 561 of

[12], with the results following easily once one has the description of the periods, provided by (3.38) of the same paper.

This Asymptotic Mirror Symmetry property is also a limiting form of the Toric Residue Mirror Conjecture of [2] (see in particular Section 9), proved in the toric complete intersection case in [13]. Once we have defined the mirror map, the limits of the Yukawa couplings may be calculated with respect to natural toric coordinates z_i at the relevant boundary point defined by the choice of framing, where $z_i = z_i(q_1, \ldots, q_r)$, since an easy application of the Chain Rule and Griffiths tranversality confirms that in the limit, the normalized Yukawa couplings coincide with the Yukawa couplings one obtains with respect to the tangent vectors $z_i \partial/\partial z_i$. An alternative approach to all this is via the Nilpotent Orbit Theorem, which yields very accurate approximations to the periods near the large complex structure limit point, and hence determines the action of monodromy on the cohomology (cf. also Theorem 5.1 of [9]).

2. AMWP metrics on Kähler moduli

Let V now denote a Calabi–Yau threefold, and σ a framing on its Kähler cone $\mathcal{K}(V)$. As in the previous section, we have a space $\mathcal{D}_\sigma$, which is biholomorphic to $(D^*)^r \subset (\mathbf{C}^*)^r$ and has $H^2(V, \mathbf{R}) + i\,\mathrm{Int}(\sigma)$ as its universal cover. The framing gives rise to coordinates $t_1, \ldots, t_r$ on $H^2(V, \mathbf{R}) + i\,\mathrm{Int}(\sigma)$, where we shall set $t_j = x_j + iy_j$, and coordinates $q_j = e^{2\pi i t_j}$ $(j = 1, \ldots, r)$ on $\mathcal{D}_\sigma$.

The asymptotic mirror symmetry property, in the form described in the previous section, leads us to considering holomorphic prepotentials $F(t_1, \ldots, t_r)$ on $H^2(V, \mathbf{R}) + i\,\mathrm{Int}(\sigma)$, of the form

$$F(t_1, \ldots, t_r) = f(t_1, \ldots, t_r)/6 + \sum a_{lm} t_l t_m + \sum b_k t_k + c + h(q_1, \ldots, q_r),$$

where f denotes the real cubic form given by cup-product, and so $f(t_1, \ldots, t_r)$ is a polynomial in the complex variables $t_1, \ldots, t_r$, and h is a holomorphic function of the q_j which vanishes at the origin (and hence, as a function of the t_j, decays exponentially as one approaches the large radius limit). Moreover, we assume that this defines a Kähler metric on some neighbourhood of the large radius limit point in $\mathcal{D}_\sigma$, with Kähler potential locally given by

$$K(t_1, \ldots, t_r) = -\log i\Big(\sum_j (t_j - \bar{t}_j)(\partial_j F + \bar{\partial}_j \bar{F}) + 2\bar{F} - 2F\Big),$$

where $\partial_j = \partial/\partial t_j$ and $\bar{\partial}_j = \partial/\partial \bar{t}_j$. We see below that this assumption forces the coefficients a_{lm}, b_k of the quadratic and linear terms to be real, but there is no such implication concerning the constant term c (cf. (4.8) of [12]).

If we know that the asymptotic mirror symmetry property holds, then we shall be interested in the case where F comes from the holomorphic prepotential defined on the mirror, and hence the Kähler metric defined above corresponds under the mirror map to the Weil–Petersson metric on some neighbourhood of the corresponding large complex structure limit point of the mirror. We shall use this in the next section to deduce interesting curvature properties of the Weil–Petersson metric for certain Calabi–Yau threefolds. In this section however, we shall restrict ourselves to considering the Kähler moduli space, equipped with a Kähler metric defined as above.

A particular special case of such a metric is when we take $F = f(t_1, \ldots, t_r)/6$, with the other terms being taken to be zero. An easy calculation (observing that we only need to check the case when f is a monomial) verifies then that

$$i\Bigg(\sum_j (t_j - \bar{t}_j)(\partial_j F + \bar{\partial}_j \bar{F}) + 2\bar{F} - 2F\Bigg) = 8f(y_1, \ldots, y_r)/6,$$

and so the metric may also be defined by taking a Kähler potential function K_0 given by $K_0(t_1, \ldots, t_r) = -\log f(y_1, \ldots, y_r)$. Since K_0 is independent of the real coordinates $x_1, \ldots, x_r$, the same will be true for the metric $(g_{i\bar{j}})$, which will be given by the formula

$$4g_{i\bar{j}} = -\partial^2(\log f)/\partial y_i \partial y_j = (\partial f/\partial y_i)(\partial f/\partial y_j)/f^2 - (\partial^2 f/\partial y_i \partial y_j)/f.$$

We see below that this is a metric on the whole complexified Kähler cone $\mathcal{K}_{\mathbf{C}}(V)$. Furthermore, if we define the *index cone* to be the open cone $W \subset H^2(V, \mathbf{R})$ where f is positive and the Hessian matrix $(\partial^2 f/\partial y_i \partial y_j)$ has index $(1, r-1)$, then the Kähler cone $\mathcal{K}(V)$ is an open subcone of W by the Hodge index theorem, and we can consider the potential function K_0 to be defined on the complexified index cone $(H^2(V, \mathbf{R}) + iW)/\mathrm{im}(H^2(V, \mathbf{Z}))$.

Lemma 2.1. *The potential function K_0 determines a Kähler metric on the whole complexified index cone, and hence on the open subset $\mathcal{K}_{\mathbf{C}}(V)$.*

Proof. Since K_0 is independent of the coordinates $x_1, \ldots, x_r$, and the corresponding matrix $g_{i\bar{j}}$ is always real symmetric, we need only demonstrate positivity. This is now the statement that we have a Riemannian metric on W defined with respect to the coordinates $y_1, \ldots, y_r$ by the matrix $-\partial^2(\log(f))/\partial y_i \partial y_j$, and that follows from [16]. □

Remark 2.2. We note for future use that a more precise result is true — see Lemma 2.4 of [25] and Theorem 1 of [16]. Note that the level set $M = W_1$ in W, given by $f = 1$, is a submanifold of W, and that the restriction of $-\partial^2 f/\partial y_i \partial y_j$ to M defines a Riemannian metric, sometimes called the *centro-affine* metric on M (this is where we use the condition on the index; see [28, 25] for further details). It is shown that the cone W equipped with the Hessian metric $-\partial^2(\log(f))/\partial y_i \partial y_j$ is isometric (up to a scaling) to the manifold $\mathbf{R} \times M$ equipped with the product metric of the standard metric on $\mathbf{R}$ with the centro-affine metric on the level set M.

For reasons which will become clear soon, we shall refer to the Kähler metric on $\mathcal{K}_{\mathbf{C}}(V)$ that we have just defined as the *Asymptotic Mirror Weil–Petersson* metric, or more concisely the AMWP metric.

Let us return now to the general case for the holomorphic prepotential F, including the quadratic, linear and constant terms, and quantum correction term h. The following lemma is then a straightforward generalization of the previous calculation

Lemma 2.3. *For the general case, the potential function $K(t_1, \ldots, t_r)$ reduces to*

$$-\log\big(8f(y_1, \ldots, y_r)/6 - 4\sum_{l,m} \mathrm{Im}(a_{lm})(x_l x_m + y_l y_m) - 4\sum_k \mathrm{Im}(b_k) x_k - 4\,\mathrm{Im}(c) + H\big),$$

where

$$H(t_1, \ldots, t_r) = \sum_j 4\pi y_j \big(\bar{q}_j(\partial \bar{h}/\partial \bar{q}_j) - q_j(\partial h/\partial q_j)\big) + 2(\bar{h} - h).$$

Proposition 2.4. *The matrix of functions $(g_{i\bar{\jmath}})$, where $g_{i\bar{\jmath}} = \partial^2 K/\partial t_i \partial \bar{t}_j$, is periodic in the real coordinates $x_1, \ldots, x_r$ if and only if the coefficients a_{lm} and b_k are all real.*

Proof. If the coefficients are real, then the $g_{i\bar{\jmath}}$ are clearly periodic, since the same is true for K. For the converse, we need to expand $g_{i\bar{\jmath}} = \partial^2 K/\partial t_i \partial \bar{t}_j$ in terms of the x_p and y_q. For any given fixed choice of values for the coordinates $y_1, \ldots, y_r$, we would obtain periodic functions in the variables $x_1, \ldots, x_r$. By inspection however, unless the coefficients $\mathrm{Im}(a_{lm})$ and $\mathrm{Im}(b_k)$ are all zero, we observe that any $g_{i\bar{\jmath}}$ may be expressed as a quotient of two functions, which for large values of the x_p are dominated by terms which are polynomial in the x_p, where the degree of the polynomial in the denominator is more than that in the numerator. This then would contradict periodicity, by considering real coordinates $x_p + n$ for n large, and observing that (for any fixed choice of values for $y_1, \ldots, y_r$) the entries in the matrix $(g_{i\bar{\jmath}})$ become arbitrarily small. □

We see therefore that we are forced to have *real* coefficients for the quadratic and linear terms in the holomorphic prepotential, and that these therefore do not contribute to the potential function K. Moreover, only the imaginary part of the constant term in the prepotential contributes to K. We may therefore replace our holomorphic prepotential F by one of the form

$$F(t_1, \ldots, t_r) = f(t_1, \ldots, t_r)/6 + i\,\mathrm{Im}(c) + h(q_1, \ldots, q_r),$$

without changing the metric. We may therefore assume that the Kähler potential is of the form $K(t_1, \ldots, t_r) = -\log\big(f(y_1, \ldots, y_r) + a + J(t_1, \ldots, t_r)\big)$, for some real constant a, and a certain function J decaying exponentially as one approaches the large radius limit. It is then clear that the corresponding matrix of partial derivatives $g_{i\bar{\jmath}} = \partial^2 K/\partial t_i \partial \bar{t}_j$ is asymptotic to the matrix defining the AMWP metric as one approaches the large radius limit, since each entry of the matrix is a quotient, the dominant terms (for large radius limit) of both the numerator and denominator being those occurring for the AMWP metric. It follows in particular that for any general choice of holomorphic prepotential F with real quadratic and linear terms, we do obtain a Kähler metric on some neighbourhood of the large radius limit point in $\mathcal{D}_\sigma$.

Lemma 2.5. *For each quadruple of indices i, j, k, l, the entry $R_{i\bar{\jmath}k\bar{l}}$ in the curvature tensor for the Kähler metric determined by a general choice of allowable holomorphic prepotential F is asymptotic to the corresponding entry of the curvature tensor in the AMWP metric.*

Proof. Given the form of the metric $g_{i\bar{\jmath}} = \partial^2 K/\partial t_i \partial \bar{t}_j$, this will follow from the formula on page 157 of [15] (valid for any Kähler metric) that

$$R_{i\bar{\jmath}k\bar{l}} = \partial^2 g_{i\bar{\jmath}}/\partial t_k \partial \bar{t}_l - \sum_{d,e} g^{\bar{e}d} (\partial g_{i\bar{e}}/\partial t_k)(\partial g_{\bar{\jmath}d}/\partial \bar{t}_l),$$

where $(g^{i\bar{\jmath}})$ is the inverse matrix to $(g_{i\bar{\jmath}})$, so that $\sum_k g^{i\bar{k}} g_{j\bar{k}} = \delta_{ij}$. The claim follows since the formula gives a sum of terms, all of which are quotients, for which the dominant terms (for large radius limit) of both the numerator and denominator are those occurring for the AMWP metric. □

Remark 2.6. In the case when the holomorphic prepotential is induced via asymptotic mirror symmetry, the AMWP metric is just what it says, and it is convenient to use this name in general. We observe that this metric is certainly independent of any choice of framing. Indeed, there is a certain amount of evidence from the toric case (see Section 4 of [11] and (4.8) of [12]) that, for holomorphic prepotentials induced via asymptotic mirror symmetry, the linear and constant terms should also be independent of the framing.

Remark 2.7. The AMWP metric is invariant under the involution given by $t_j \to -\bar{t}_j$, which implies that the submanifold $\mathcal{K}(V)$ of $\mathcal{K}_{\mathbf{C}}(V)$ is the fixed locus of an isometric involution, and hence is a totally geodesic submanifold (see [14], page 59). In particular, for any tangent plane to $\mathcal{K}(V)$, the sectional curvature of the Kähler metric on $\mathcal{K}_{\mathbf{C}}(V)$ is the same as the sectional curvature of the restricted metric on $\mathcal{K}(V)$, which we noted above is (up to scaling) the product of the standard metric on $\mathbf{R}$ with the induced (centro-affine) metric on the level set $\mathcal{K}_1$ given by $f = 1$. In the case when the holomorphic prepotential is induced via asymptotic mirror symmetry, the curvature of the level set $\mathcal{K}_1$ reflects the asymptotic behaviour of an appropriate slice of the Weil–Petersson metric near the corresponding large complex structure limit point for the mirror. This was a claim made in Section 1 of [28], but the justification just given is somewhat more convincing than that given in [28].

In the case when asymptotic mirror symmetry applies, we may use it to prove a formula for the curvature of AMWP metric, via Strominger's formula for the curvature of the Weil–Petersson metric on the complex moduli space of the mirror. We denote the AMWP metric, with Kähler potential $-\log f(y_1, \dots, y_r)$, by the matrix $(g_{i\bar{j}})$ (with respect to the coordinates $t_j = x_j + iy_j$), and let f_{ijk} denote the third partial derivatives of the cubic form f. On the mirror side, there is a well-known formula, re-proved by Strominger, for the curvature of the Weil–Petersson metric [22, 17, 21, 26]. The $(\Omega, \bar{\Omega})$ term in the denominator of Strominger's formula corresponds asymptotically (as checked above) to $8f(y_1, \dots, y_r)/6$, and so the formula predicts an analogous formula for the curvature tensor of the AMWP metric $g_{i\bar{j}}$.

Conjecture 2.8. *The curvature tensor is given by the formula*

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}} - \sum_{p,q} \frac{g^{p\bar{q}} f_{ikp} f_{jlq}}{64\, f^2},$$

where $(g^{p\bar{q}})$ denotes the inverse matrix to $(g_{i\bar{j}})$.

This conjecture is an algebraic identity, which should therefore hold for arbitrary cubics f, on their complexified index cone. We believe that it follows in general from the theory of projective special Kähler manifolds [7], but in this paper we shall only need it for a general ternary cubic, i.e. when $r = 3$, for which a more elementary proof suffices.

Lemma 2.9. *Suppose f is a ternary cubic, then the above formula for the curvature tensor of the AMWP metric on the complexified index cone holds.*

Proof. Suppose that r is arbitrary and we make a linear change of variables $y_i = \sum a_{ij} y'_j$, with $A = (a_{ij})$ a real invertible matrix; this corresponds to a linear change of variables in the complex coordinates $t_i = \sum a_{ij} t'_j$. Therefore $\partial t_e / \partial t'_j = a_{ej}$ is real

for all e, j. We observe now that the components of the curvature tensor transform via

$$R'_{i\bar{j}k\bar{l}} = \sum a_{di} a_{ej} a_{mk} a_{nl} R_{d\bar{e}m\bar{n}}.$$

We claim that the right-hand side of the desired identity transforms in the same way. For this we note that

$$g'_{p'q'} = -\frac{\partial}{\partial t'_{p'}} \frac{\partial}{\partial \bar{t}'_{q'}} \log f(y_1, y_2, y_3) = \sum \frac{\partial t_p}{\partial t'_{p'}} \frac{\partial \bar{t}_q}{\partial \bar{t}'_{q'}} g_{p\bar{q}} = \sum a_{pp'} a_{qq'} g_{p\bar{q}},$$

or in matrix notation that $g' = A^t g A$, and hence $(g')^{-1} = A^{-1} g^{-1} (A^t)^{-1}$. It follows therefore that for any holomorphic functions u and v,

$$\sum (g')^{p'\bar{q}'} \frac{\partial u}{\partial t'_{p'}} \frac{\partial \bar{v}}{\partial \bar{t}'_{q'}} = \sum g^{pq} \frac{\partial u}{\partial t_p} \frac{\partial \bar{v}}{\partial \bar{t}_q}.$$

Applying this to $u = \partial^2 f / \partial t'_i \partial t'_k$ and $v = \partial^2 f / \partial t'_j \partial t'_l$, where $f = f(t_1, \ldots, t_r)$ here denotes the cubic considered as a polynomial function in the complex variables, and expressing u and v in terms of the $\partial^2 f / \partial t_e \partial t_n$, we deduce that the right-hand side of the identity transforms as claimed.

We now specialise to $r = 3$; in order to prove the desired identity for ternary cubics, we may therefore assume that the real cubic $f(y_1, y_2, y_3)$ is in an appropriate form, say Weierstrass canonical form $f(y_1, y_2, y_3) = y_2^2 y_3 - y_1^3 - \lambda y_1 y_3^2 - \mu y_3^3$. The validity of the identity in this case may be checked by hand (or more conveniently by computer). □

If we know Conjecture 2.8 holds in a given case, it has the consequence that the holomorphic bisectional curvatures are bounded below by a fixed constant, namely -2; to see this, we need only observe that for fixed i, j, the real symmetric matrix $(f_{ijp} f_{ijq})$ is positive semi-definite. This in turn implies that both the holomorphic sectional curvatures and the Ricci curvatures are bounded below (using Section 7.5 of [30]), and hence the same is true for the scalar curvature (explicitly, the holomorphic sectional curvatures are bounded below by -2, the Ricci curvatures by $-(r+1)$ and the scalar curvature by $-r(r+1)$); these statements are exactly analogous to the ones made concerning the Weil–Petersson metric in [21, 26].

3. Weil–Petersson curvature near large complex structure limits

If we know that the asymptotic mirror symmetry property holds for a pair of Calabi–Yau threefolds V° and V, then the asymptotic behaviour of the Weil–Petersson metric near large complex structure limit for V° will be encoded by the behaviour of the AMWP metric on the complexified Kähler cone $\mathcal{K}_{\mathbf{C}}(V)$ for V. In Remark 2.7, we observed that for tangent planes to the submanifold $\mathcal{K}(V)$, the curvature of the AMWP metric is just that of the restricted metric on $\mathcal{K}(V)$, which (up to a constant scaling) is the Hessian metric corresponding to the function $-\log f(y_1, \ldots, y_r)$, where f denotes the cup-product cubic form. However, in Remark 2.2, we commented that this Hessian metric is (up to scaling) isometric to the metric on $\mathbf{R} \times \mathcal{K}_1$ given by the product of the standard metric on $\mathbf{R}$ with the centro-affine metric on $\mathcal{K}_1$ defined by the restriction of the negative Hessian matrix $(-\partial^2 f / \partial y_i \partial y_j)$. Modulo a factor of 1/6, this latter metric is the one studied in [28].

We work now under the assumption that the holomorphic bisectional curvatures of the AMWP metric are bounded below, and hence that the same also holds for the holomorphic sectional curvatures. This holds for instance whenever Conjecture

2.8 is true, which has been shown above to be the case for arbitrary cubics when $r = 3$, and for cubics given by cup-product on Calabi–Yau threefolds when we know that asymptotic mirror symmetry does apply. We remark also that, for any Kähler metric, if the holomorphic sectional curvatures are bounded absolutely, then so too are the sectional curvatures, since the sectional curvatures may be expressed in terms of the holomorphic sectional curvatures (Lemma 7.19 of [30]).

Proposition 3.1. *If the holomorphic bisectional curvatures of the AMWP metric on $\mathcal{K}_{\mathbf{C}}$ are bounded below, and the centro-affine metric on $\mathcal{K}_1$ has sectional curvatures which are unbounded above, then the AMWP metric has Ricci curvatures which are unbounded above. The scalar curvature will then also be unbounded above.*

Proof. From the facts noted above, the holomorphic sectional curvatures of the AMWP metric must also take arbitrarily large values. Since the Ricci curvature is a sum of holomorphic bisectional curvatures ([30], page 180), the assumptions imply that it too is bounded below by some fixed value, but one can find values which are arbitrarily large positive (using the unboundedness of the holomorphic sectional curvatures). Since the scalar curvature is a sum of Ricci curvatures, this confirms that the scalar curvature also takes arbitrarily large positive values on $\mathcal{K}_{\mathbf{C}}(V)$. □

We now restrict ourselves to the case when $r = 3$. Here, the level set $\mathcal{K}_1$ is a surface, and a formula was proved in [28] for the Gaussian curvature R of the centro-affine metric scaled by $1/6$ (which was called the *Hodge metric* on $\mathcal{K}_1$ there), which may be expressed as

$$R = -\frac{9}{4} + \frac{S}{4\,h^2},$$

where S is the S-invariant of the cubic f, and $h = \det(\frac{1}{6}\partial^2 f/\partial y_i \partial y_j)$, that is the Hessian determinant of f, scaled by 6^{-3}. The S-invariant of a ternary cubic is decribed in [29, 28, 25]. From the formula it follows that if $S > 0$ and there is a ray in the boundary of the Kähler cone on which the cubic f does not vanish but the Hessian determinant does, then the Gaussian curvature of the centro-affine metric is unbounded above, from which it follows as above that the AMWP metric has scalar curvature unbounded above on $\mathcal{K}_{\mathbf{C}}$.

It was shown in [27] that the boundary of the Kähler cone is locally polyhedral away from the cone given by $f = 0$ (known in [27] as the cubic cone). The codimension one faces so obtained correspond to primitive birational contractions of the threefold, and were classified into three types: Type I, where only finitely many (rational) curves are contracted; Type II, where a surface (a generalised del Pezzo surface) is contracted to a point; Type III, where a surface E is contracted to a smooth rational curve C, with E being a conic bundle over C. For rays in the interior of a Type II face, it follows immediately that the cubic f does not vanish, but the Hessian determinant does. Here, the cubic is of the form $u_1^3 + g(u_2, u_3)$ with respect to suitable real coordinates u_1, u_2, u_3, from which it follows that the S-invariant $S = 0$. We shall study this case in Theorem 3.7 below. It should be remarked that for all known examples of Calabi–Yau threefolds with $r = 3$, the S-invariant of the cubic form is non-negative.

Another case when one has a ray along which the cubic is non-vanishing but the Hessian determinant is zero is given by a codimension *two* face corresponding to the contraction of a surface to a point. This face will be the intersection of faces of Types

I or III, and will not automatically imply that $S = 0$. Examples of such Calabi–Yau threefolds with $h^{1,1} = 3$ are given by smooth Calabi–Yau Weierstrass models over smooth surfaces Y, where it is shown in [19] that Y is one of the Hirzebruch rational surfaces $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$, $\mathbf{F}_1$ or $\mathbf{F}_2$. These Calabi–Yau threefolds all have Picard groups generated by the class of the section $E \cong Y$ of the fibration and the classes pulled back from $\mathrm{Pic}(Y)$ (generating a hyperplane in $\mathrm{Pic}(V) \otimes \mathbf{R}$). Here, there will be a codimension two face of the Kähler cone corresponding to the contraction of E. In all three cases, the cubic form is reducible, defining a real plane projective cubic consisting of a line (given by the classes pulled back from the base) and an irreducible conic intersecting the line in two points; the cubic then has S-invariant $S > 0$, as desired. For such a Calabi–Yau threefold, Proposition 3.1 ensures that the scalar curvature of the AMWP metric is unbounded above; in Example 3.2 below, we shall see this explicitly in the third of the above cases. The Kähler cone is simplicial, and the codimension two face of the Kähler cone corresponding to the contraction of the section E is the codimension two face which is not contained in the hyperplane of classes pulled back from the base. In the three cases, the ray is the intersection of faces of Types III and III , Types I and III, and Types III and III, respectively. In the first case, the faces both correspond to contractions of E (along different rulings), whilst in the third case, there are two different exceptional surfaces E and D, which intersect along the minimal section of $E \cong \mathbf{F}_2$.

Example 3.2. Let us concentrate on the case above of the Calabi–Yau threefolds V which are Weierstrass fibrations over $\mathbf{F}_2$. We shall denote the section of the Weierstrass fibration by E, the pullback of a fibre of the ruling on the base by L and the pullback of the (-2)-curve on the base by D. These classes generate the Picard group of V, and the non-zero intersection numbers are given by $E^3 = 8$, $E^2 \cdot L = -2$, $E \cdot D^2 = -2$ and $E \cdot D \cdot L = 1$. The generators of the Kähler cone are then checked to be given by $J_1 = E + 2D + 4L$, $J_2 = L$ and $J_3 = D + 2L$. This leads to the following cubic intersection form:

$$f = 8y_1^3 + 12y_1^2 y_3 + 6y_1^2 y_2 + 6y_1 y_3^2 + 6y_1 y_2 y_3,$$

where the Kähler cone is given by $y_1, y_2, y_3 \geq 0$. The S-invariant of this ternary cubic form takes the value 1. In passing, we note that the classes J_1, J_2, J_3 generate $H^2(V, \mathbf{Z})$, and so there is an obvious natural framing for the Kähler cone.

The AMWP metric is given by specifying $-\log f(y_1, y_2, y_3)$ to be the Kähler potential. For the determinant of the metric we then obtain (compare also with Lemma 3.5 below)

$$\det(g_{k\bar{l}}) = \frac{27(y_1y_2 + 2y_1y_3 + y_3^2 + y_2y_3)}{64\, y_1^2(3y_2y_3 + 4y_1^2 + 3y_1y_2 + 6y_1y_3 + 3y_3^2)^3}.$$

Using the fact that the Ricci tensor of any Kähler metric is given by the formula (see [15], page 158)

$$R_{i\bar{j}} = -\partial^2(\log(\det(g_{k\bar{l}}))/\partial t_i \partial \bar{t}_j,$$

we can calculate (using MATHEMATICA) that the scalar curvature is given by

$$\begin{gathered}\tfrac{2}{3(y_3(y_2+y_3)+y_1(y_2+2y_3))^3}\big(16y_1^6 - 9y_3^3(y_2+y_3)^3 + 24y_1^5(y_2+2y_3) - \\ 27y_1y_3^2(y_2+y_3)^2(y_2+2y_3) + 12y_1^4(y_2^2 + 6y_2y_3 + 6y_3^2) - \\ 3y_1^3(3y_2^3 + 10y_2^2y_3 + 12y_2y_3^2 + 8y_3^3) - 3y_1^2y_3(9y_2^3 + 41y_2^2y_3 + 64y_2y_3^2 + 32y_3^3)\big)\end{gathered}$$

We can now see explicitly that the scalar curvature blows up to $+\infty$ if we approach the large radius limit along for instance the curve in $\{\mathbf{0}\} \times i\mathcal{K}(V)$ given by $(y_1, y_2, y_3) = (s^2, s, s)$, letting $s \to \infty$. It should be noted that using a curve of the form (s, as, bs) is not sufficient to get a blowing up of the scalar curvature, since along such a line it is independent of s.

There is however another useful description of this Calabi–Yau threefold V as the resolution of a general hypersurface of degree 24 in weighted projective space $\mathbf{P}(1,1,2,8,12)$; one such hypersurface would be of Fermat type with equation

$$z_0^{24} + z_1^{24} + z_2^{12} + z_3^3 + z_4^2 = 0.$$

The general hypersurface inherits singularities from the singular locus of the ambient space, namely it contains an elliptic curve C of $\mathbf{Z}_2$ quotient singularities, which in turn contains an exceptional $\mathbf{Z}_4$ quotient singularity (see Section 2 of [10]). Resolving singularities, we obtain a surface $D \cong C \times \mathbf{P}^1$ which contracts down to C, and a surface E isomorphic to $\mathbf{F}_2$ which contracts down to the $\mathbf{Z}_4$ point on C; the intersection of these two surfaces is a curve which is both the minimal section of E and a fibre of the ruling on D. The resolution may be checked to be a Weierstrass fibration over $\mathbf{F}_2$, where E and D have the same meanings as before. It is noted in Section 3 of [18] that such a Calabi–Yau threefold has Hodge numbers $h^{1,1} = 3$ and $h^{1,2} = 243$, but that only 242 dimensions of the complex moduli are realised as polynomial deformations.

Finally, one should remark that this threefold is often referred to in the Physics literature as the STU-model; when one makes the change of coordinates $U = y_1$, $S = y_2$ and $y_3 = T - U$, the cubic $f/6$ takes the form $STU + T^2U + \frac{1}{3}U^3$.

We are now in a position to disprove the conjecture made in [20] concerning the non-positivity of the scalar curvature of the Weil–Petersson metric in some neighbourhood of the large complex structure limit point.

Theorem 3.3. *If we consider the mirror V° to the resolution V of a general hypersurface of degree* 24 *in* $\mathbf{P}(1,1,2,8,12)$*, with the natural framing on* $\mathcal{K}(V)$*, then there is a corresponding large complex structure limit point in the moduli space of* V°*, and in any neighbourhood of this boundary point the scalar curvature of the Weil–Petersson metric is unbounded above.*

Proof. Modulo a few mechanical checks, this follows by combining the various results we have proved above. The weighted projective space $\mathbf{P}(1,1,2,8,12) = \mathbf{P}(\Delta)$, for Δ the polytope with vertices.

$$\begin{aligned} v_1 &= (1,-1,-1,-1) \\ v_2 &= (-1,2,-1,-1) \\ v_3 &= (-1,-1,11,-1) \\ v_4 &= (-1,-1,-1,23) \\ v_5 &= (-1,-1,-1,-1). \end{aligned}$$

For the polar polytope Δ°, we have vertices:

$$\begin{aligned}
v_1^* &= (1,0,0,0)\\
v_2^* &= (0,1,0,0)\\
v_3^* &= (0,0,1,0)\\
v_4^* &= (0,0,0,1)\\
v_5^* &= (-12,-8,-2,-1).
\end{aligned}$$

The other integral points (apart from the origin) of Δ° are

$$\begin{aligned}
v_6^* &= (-3,-2,0,0) = \frac{1}{2}v_3^* + \frac{1}{4}v_4^* + \frac{1}{4}v_5^*\\
v_7^* &= (-6,-4,-1,0) = \frac{1}{2}v_4^* + \frac{1}{2}v_5^*\\
v_8^* &= (-1,-1,0,0) = \frac{1}{2}v_1^* + \frac{1}{4}v_3^* + \frac{1}{8}v_4^* + \frac{1}{8}v_5^*\\
v_9^* &= (-2,-1,0,0) = \frac{1}{3}v_2^* + \frac{1}{3}v_3^* + \frac{1}{6}v_4^* + \frac{1}{6}v_5^*\\
v_{10}^* &= (-1,0,0,0) = \frac{1}{2}v_9^* + \frac{1}{2}v_2^*.
\end{aligned}$$

Note that we have one interior point of a codimension 2 face of Δ°, namely v_6^*. The corresponding dual face of Δ is spanned by v_1 and v_2, and there are no integral points in the interior of this face. Thus (unlike the case of V), the complex deformations of V° all arise from deformations of the defining polynomial. It is readily checked that the fan given by cones on the faces of Δ° with vertices at the origin may be subdivided at the extra integral points $v_6^*, v_7^*, v_8^*, v_9^*, v_{10}^*$ so as to achieve a regular fan. Because of the existence of interior points v_8^*, v_9^*, v_{10}^* of codimension one faces, in the terminology of [11, 12] the polytope Δ° is of Type II rather than Type I.

These calculations ensure that there is a large complex structure limit point for V° corresponding to the natural framing on the Kähler cone of V, and that the asymptotic toric mirror symmetry property does hold. Thus the asymptotic form the Weil–Petersson metric and its scalar curvature near large complex structure limit for V° will be encoded by the AMWP metric and its scalar curvature on $\mathcal{K}_{\mathbf{C}}(V)$; as we saw in the previous calculations the scalar curvature is unbounded above. □

Remark 3.4. Suppose now we choose a framing σ of the above Kähler cone whose interior is contained in the open subcone of $\mathcal{K}(V)$ where the AMWP metric has positive scalar curvature. Such a framing will correspond to a large complex structure point of the mirror (on a different compactification of the moduli space) with the property that, in some open neighbourhood of this boundary point, the Weil–Petersson metric has everywhere positive scalar curvature.

By studying the cubic intersection forms for toric hypersurface Calabi–Yau threefolds listed in Appendix C of [12], we find other candidates with $h^{1,1} = 3$ where the above theory holds in exactly the same way. The reader should be aware that the way these forms are tabulated in the Physics literature omits binomial coefficients, and so what appears as $8J_1^3 + 2J_1^2J_2 + 4J_1^2J_3 + J_1J_2J_3 + 2J_1J_3^2$ in Appendix C of

[12] is exactly the cubic form for the STU-model calculated above. For all the examples listed there, the Kähler cone is simplicial with generators J_1, J_2 and J_3. For those examples which are of Types I or II, we can check which intersection forms have strictly positive S-invariant and which models have codimension two faces (i.e. rays) in the boundary of the Kähler cone along which the Hessian determinant vanishes but the cubic form is non-zero. Apart from $V_{24} \subset \mathbf{P}(1,1,2,8,12)$, we find further examples $V_{10} \subset \mathbf{P}(1,1,2,2,4)$, $V_{16} \subset \mathbf{P}(1,1,3,3,8)$, $V_{18} \subset \mathbf{P}(1,2,2,4,9)$ and $V_9 \subset \mathbf{P}(1,1,1,2,4)$, where a precisely analogous statement to Theorem 3.3 will hold. As remarked however in Section 5 of [12], the last two of these are in fact isomorphic.

When the S-invariant is zero, we do not have any blowing up of sectional curvatures of the AMWP metric on tangent planes to the slice $i\mathcal{K}(V)$ in $\mathcal{K}_{\mathbf{C}}(V)$; the scalar curvature may still however be unbounded above. To see why this is, let us continue to restrict ourselves to the case of $r = 3$, and let

$$g_{i\bar{j}} = \big((\partial f/\partial y_i)(\partial f/\partial y_j) - f\,\partial^2 f/\partial y_i \partial y_j\big)/4f^2$$

denote the AMWP metric.

Lemma 3.5. *If G denotes the matrix with entries*

$$G_{ij} = (\partial f/\partial y_i)(\partial f/\partial y_j) - f\,\partial^2 f/\partial y_i\partial y_j = -f^2 \partial^2(\log f)/\partial y_i \partial y_j,$$

then $\det G = \frac{1}{2} f^3 H$.

Proof. Under linear changes of variables $y_i = \sum a_{ij} y_j'$, with $A = (a_{ij})$ a real invertible matrix, we note that the matrix G transforms to $A^t G A$ and the Hessian matrix of f transforms similarly. Hence both $\det G$ and the Hessian determinant H transform via multiplication by $(\det A)^2$. Therefore, in order to prove the desired identity, we may assume that f is in an appropriate canonical form, say Weierstrass canonical form $f = y_2^2\, y_3 - y_1^3 - \lambda y_1 y_3^2 - \mu y_3^3$. The validity of the identity in this case may be checked by hand (or computer). □

Let us now set $B = \mathrm{Adj}\,(G)$, the matrix of cofactors of G, and let $C(i,j)$ denote the rank 1 positive semi-definite matrix with $C(i,j)_{pq} = f_{ijp} f_{ijq}$. From the formula proved in Lemma 2.9 for the curvature of the AMWP metric, we may deduce the following criterion for the unboundedness of the scalar curvature.

Lemma 3.6. *If there exists a ray in the boundary of the Kähler cone along which the cubic f does not vanish, the Hessian determinant H does vanish, and for which* $\mathrm{tr}(B\,C(i,j))$ *does not vanish for some i, j, then the scalar curvature of the AMWP metric is unbounded above as one approaches the given ray.*

Proof. Using Lemma 2.9, we consider the formula for $R_{i\bar{i}j\bar{j}}$; using the above lemma, the final term in the formula reduces to $-\frac{1}{8}\mathrm{tr}(B\,C(i,j))/Hf^3$. Our assumptions therefore imply that $-R_{i\bar{i}j\bar{j}}$ is unbounded above, and hence the same is true for the corresponding holomorphic bisectional curvature. Since the holomorphic bisectional curvatures are bounded below, it follows immediately that some Ricci curvature is unbounded above, and hence the same is true for the scalar curvature. □

We now return as promised to the case where some codimension one face of the Kähler cone corresponding to a Type II contraction.

Theorem 3.7. *If V is a Calabi–Yau threefold with $h^{1,1} = 3$, for which there is a face of the Kähler cone determining a contraction of Type II, then the scalar curvature of the AMWP metric on $\mathcal{K}_{\mathbf{C}}(V)$ is unbounded above.*

Proof. By choosing appropriate linear coordinates, we may assume that the cubic f takes the form $f = y_1^3 + y_2(ay_2^2 + 3by_2y_3 + 3cy_3^2)$, with a, b, c real coefficients, not both of b and c being zero, and where the hyperplane defining the Type II face is $y_1 = 0$. In the proof of Lemma 2.9, we saw that the formula for the entries of the curvature tensor remained valid whatever linear coordinates were used. We therefore use the above coordinates. With the notation as in Lemma 3.5, we set $C = C(1,1)$, and calculate that $\mathrm{tr}(BC)$ is a non-zero constant multiple of

$$((b^2 - ac)y_2^2 + bc\, y_2y_3 + c^2y_3^2)(f - 3y_1^3)f,$$

which is non-vanishing at the general point of the hyperplane $y_1 = 0$. In particular, it follows from Lemma 3.6 that the scalar curvature of the AMWP metric is unbounded above. □

Example 3.8. Consider a Calabi–Yau threefold V from [12] given as a resolution of a hypersurface $V_{16} \subset \mathbf{P}(1,1,1,5,8)$. It is shown in [12] that this is of Type II, and so the asymptotic toric mirror symmetry property holds, and $h^{1,1} = 3$. The topological cubic form is

$$f = 50y_1^3 + 30y_1^2y_2 + 6y_1y_2^2 + 240y_1^2y_3 + 96y_1y_2y_3 + 9y_2^2y_3 + 384y_1y_3^2 + 75y_2y_3^2 + 203y_3^3.$$

The Hessian of this vanishes along the hyperplane $y_2 = 0$, and this hyperplane defines a contraction of V of Type II. In particular, the S-invariant is zero. Theorem 3.7 shows that the scalar curvature of the AMWP metric on $\mathcal{K}_{\mathbf{C}}(V)$ is unbounded above, and hence, by asymptotic mirror symmetry, the scalar curvature of the Weil–Petersson metric is unbounded above in any neighbourhood of the corresponding large complex structure limit point of the complex moduli space of the mirror.

Similar statements are true for the Calabi–Yau threefold arising from $V_{12} \subset \mathbf{P}(1,1,1,3,6)$, which is also of Type II (and so the asymptotic toric mirror symmetry property holds) and has $h^{1,1} = 3$. The topological cubic form is

$$f = 18y_1^3 + 18y_1^2y_2 + 54y_1^2y_3 + 6y_1y_2^2 + 36y_1y_2y_3 + 54y_1y_3^2 + 3y_2^2y_3 + 9y_2y_3^2 + 9y_3^2,$$

and the Hessian vanishes along the codimension one face of the Kähler cone given by $y_2 = 0$.

In a similar way, certain toric hypersurface Calabi–Yau threefolds admitting a Type II contraction and with $h^{1,1} = 2$ also provide counterexamples to the original conjecture.

References

[1] Batyrev, V.V.: Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties. J. Alg. Geom. **3**, 493-535 (1994)

[2] Batyrev, V.V., Materov, E.N.: Toric Residues and Mirror Symmetry, Mosc. Math. J. **2**, 435-475 (2002)

[3] Bertin, J., Peters, Ch.: Variations de structures de Hodge, variétés de Calabi–Yau et symétrie miroir. In: Introduction à la théorie de Hodge (J. Bertin, J.-P. Demailly, L. Illusie, Ch. Peters), pp 169-256. Société Mathématique de France, 1996

[4] Bryant, R., Griffiths, P.: Some observations on the infinitesimal period relations for regular threefolds with trivial canonical bundle, Arithmetic and Geometry vol. II, M. Artin and J. Tate (eds.), Progress in Math. vol. 36, Birkhäuser, Boston, 1983, 77-102.

[5] Candelas, P., De la Ossa, X. C., Green, P. S., Parkes, L.: An exactly soluble superconformal theory from a mirror pair of Calabi-Yau manifolds, Phys. Lett. B. **258**, 118–126 (1991)
[6] Cox, D., Katz, S.: Mirror Symmetry and Algebraic Geometry. Mathematical Surveys and Monographs, **68**. AMS Publishing, 1999
[7] Freed, D.S.: Special Kähler manifolds, Comm. Math. Phys. **203**, 31-52 (1999)
[8] Givental, A.: A mirror theorem for toric complete intersections. In: Topological field theory, primitive forms and related topics (eds Kashiwara, M. et al.). Proceedings of the 38th Taniguchi symposium, Kyoto, Japan, December 9-13, 1996. Birkhäuser Prog. Math. **160**, 141-175 (1998)
[9] Gross, Mark; Siebert, Bernd: Mirror Symmetry via logarithmic degeneration data II. arXiv:07092290
[10] Hosono, S., Klemm A., Theisen S., Yau, S.-T.: Mirror Symmetry, Mirror Map and Applications to Calabi-Yau Hypersurfaces, Comm. Math. Phys. **167**, 301-350 (1995)
[11] Hosono, S., Klemm A., Theisen S., Yau, S.-T.: Mirror Symmetry, Mirror Map and Applications to Complete Intersection Calabi-Yau Spaces, Nuclear Physics B. **433**, 501-552 (1995)
[12] Hosono, S., Lian, B.H., Yau, S.T.: GKZ-Generalized Hypergeometric Systems in Mirror Symmetry of Calabi–Yau Hypersurfaces. Commun. Math. Phys. **182**, 535-577 (1996)
[13] Karu, K.: Toric residue mirror conjecture for Calabi–Yau complete intersections, J. Algebraic Geom. **14**, 741-760 (2005)
[14] Kobayashi, S.: Transformation Groups in Differential Geometry. Berlin-Heidelberg: Springer 1985
[15] Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, Volume II. John Wiley and Sons, 1969
[16] Loftin, J.C.: Affine Spheres and Kähler-Einstein Metrics, Math. Res. Letters, **9**, 425–432 (2002)
[17] Lu, Z., Sun, X.: Weil-Petersson Geometry on the Moduli Space of Polarized Calabi-Yau Manifolds, J. Inst. Math. Jussieu, **3**, 185–229 (2004)
[18] Morrison, D., Vafa, C.: Compactifications of F-Theory on Calabi–Yau Threefolds – I, Nuclear Physics B. **473**, 74–92 (1996)
[19] Nakayama, N.: On Weierstrass models. In: Algebraic geometry and commutative algebra, Vol.II, pp 405-431. Kinokuniya, Tokyo, 1988
[20] Ooguri, H., Vafa, C.: On the geometry of the string landscape and the swampland, Nuclear Physics B. **766**, 21–33 (2007)
[21] Schumacher, G.: The curvature of the Petersson–Weil metric on the moduli space of Kähler–Einstein manifolds. In: Complex Analysis and Geometry (eds V. Ancona & A. Silva), pp 339-354. Plenum Press, New York 1993
[22] Strominger, A.: Special Geometry, Comm. Math. Phys. **133**, 163–180 (1990)
[23] Tian, G.: Smoothness of the Universal Deformation Space of Compact Calabi-Yau Manifolds and its Petersson-Weil metric. In S.-T. Yau, ed., Mathematical aspects of string theory, vol. 1, 629–646. World Scientific, 1987
[24] Todorov, A.N.: The Weil-Petersson geometry of the moduli space of $SU(n \leq 3)$ (Calabi-Yau) manifolds. I. Comm. Math. Phys **126**, 325–346 (1989)
[25] Totaro, B.: The Curvature of a Hessian Metric. Int. J. Math. **15**, 369-391 (2004)
[26] Wang, C.-L: Curvature Properties of The Calabi-Yau Moduli, Documenta Mathematica **8**, 577–590 (2003)
[27] Wilson, P.M.H.: The Kähler cone on Calabi–Yau threefolds, Invent. math. **107** (1992), 561-583; Erratum: Invent. math. **114**, 231-232 (1994)
[28] Wilson, P.M.H.: Sectional Curvatures of Kähler moduli, Math. Ann. **330**, 631–664 (2004)
[29] Wilson, P.M.H.: Some Geometry and Combinatorics for the S-Invariant of Ternary Cubics, Experiment. Math. **15**, 479–490 (2006)
[30] Zheng, F.: Complex Differential Geometry. Studies in Advanced Mathematics **18**. AMS Publishing and International Press, 2000.

Department of Pure Mathematics, University of Cambridge, 16 Wilberforce Road, Cambridge CB3 0WB, UK

E-mail address: t.trenner@dpmms.cam.ac.uk, pmhw@dpmms.cam.ac.uk